

Strong NP-hardness of AC power flows feasibility

Daniel Bienstock and Abhinav Verma, Columbia University

1 Introduction

The AC-OPF problem has lately received increased attention, spurred by the work in [7]. An important question concerns the fundamental complexity of AC-OPF. In this setting we note [8], which discusses a proof of weak NP-hardness of AC-OPF on *trees*; also see [7] for an outline of a proof.

The purpose of this note is to present a rigorous proof of strong NP-hardness of AC-OPF on general graphs; this proof builds on a section in the PhD thesis [10]. A challenge in the development of such a proof is the fact that the solution to an AC-OPF problem may have irrational coordinates; this necessitates an elaboration in our technique.

The problem we consider is described by a directed graph G representing a power transmission system, where for each line (i.e., arc) (i, j) we are given a positive parameter x_{ij} , the *reactance* and a nonnegative value θ_{ij}^{\max} , the *maximum phase angle difference*. Additionally for each bus (i.e., vertex) i we have a value b_i indicating the net generation at i . We assume $\sum_i b_i = 0$. Using standard network flows formulation, the lossless AC power flow feasibility system on G can be written as

$$Nf = b, \quad (1a)$$

$$\sin(\theta_i - \theta_j) = x_{ij} f_{ij}, \quad \forall (i, j), \quad (1b)$$

$$|\theta_i - \theta_j| \leq \theta_{ij}^{\max}, \quad \forall (i, j). \quad (1c)$$

Typically, $\theta_{ij}^{\max} < \pi/2$ for each line (i, j) . System (1) is an approximation to more comprehensive AC power flow models; given a line (i, j) variable f_{ij} models the real (active) power flowing from i to j , which could be negative, and for any bus i variable θ_i is the phase angle at i . See [1], [3], [6]. This model, and variants thereof, was considered in [9], [5], [2]. From a power engineering perspective the model assumes zero resistances and unconstrained reactive power flows and injections.

System (1) is a nonconvex system of equations. In this section we prove that testing feasibility of such a system is a strongly NP-hard problem, that is to say it remains NP-hard even if the number of bits in the input data is polynomially bounded as a function of the number of buses. We also point out [4], which shows that AC-OPF problems on graphs with bounded tree-width can be approximated arbitrarily closely in polynomial time.

1.1 Main construction

Define

$$\Delta(\theta) \doteq -\sin(\theta) + \frac{5}{8}\sin(2\theta),$$

and set

$$\theta_0 = \cos^{-1}\left(\frac{4}{5}\right) \approx .6435, \quad \theta_1 = \cos^{-1}\left(\frac{1}{5} + \sqrt{\frac{1}{25} + \frac{1}{2}}\right) \approx .3630.$$

Then we have:

Lemma 1 Suppose $0 \leq \theta \leq \frac{\pi}{2}$. Then:

(a) $\Delta(\theta) = 0$ iff $\theta = 0$ or $\theta = \theta_0$.

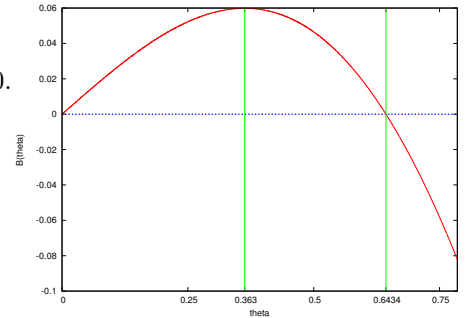


Figure 1: $\Delta(\theta)$ for $\theta \in [0, \pi/4]$

(b) $\Delta'(\theta) > 0$ for $0 \leq \theta < \theta_1$, $\Delta'(\theta_1) = 0$ and $\Delta'(\theta) < 0$ for $\theta_1 < \theta \leq \pi/2$.

Proof. Part (a) is clear. To prove (b) let $c = \cos(\theta)$. Then $\Delta'(\theta) = -c + \frac{5}{4}(2c^2 - 1)$ whose only zero in $[0, 1]$ is at $1/5 + \sqrt{1/25 + 1/2}$. ■

See Figure 1.

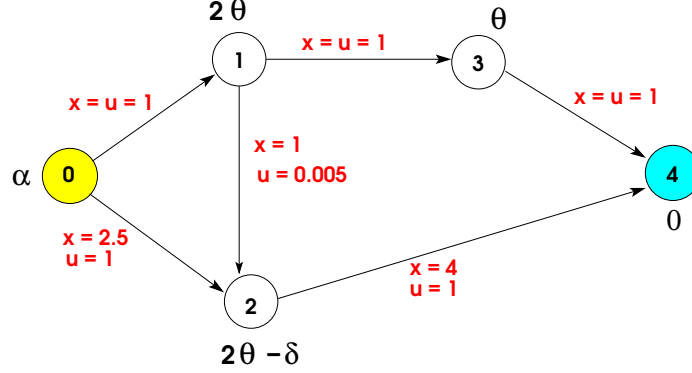


Figure 2: Basic transmission system B used in proof

Our construction is centered on the transmission system B shown in Figure 2. In this figure we show, next to each line, its reactance (“x”) and its limit (“u”). We consider solutions to the lossless AC model where bus 0 injects power into the system, bus 4 withdraws power, and all other buses have zero net balance. The figure also shows our naming convention for phase angles in a solution to the lossless AC power flow problem on this graph. Denoting the phase angle at bus i by θ_i , for $0 \leq i \leq 4$, we have:

- (i) Without loss of generality $\theta_4 = 0$,
- (ii) We simplify θ_3 as θ .
- (iii) $\theta_1 = 2\theta$ (flow conservation at bus 3).
- (iv) We write θ_2 as $2\theta - \delta$. Possibly $\delta < 0$.
- (v) We write $\theta_0 = \alpha$.

We can make some basic observations:

1. Since power flows from 3 to 4,

$$0 \leq \theta \leq \pi/2. \quad (2)$$

2. The absolute value of the flow on line (1, 2) is $|\sin(\delta)|$. But the flow limit on line (1, 2) is 0.005, implying

$$|\sin \delta| \leq 0.005, \quad \text{and consequently} \quad |\delta| < 0.0050001. \quad (3)$$

Remark. The bounds in (3) can be made arbitrarily small by choosing a small enough limit on line (1,2).

3. Flow on (0,1) and (0,2) must be nonnegative. Together with the phase angle limits we have

$$\max\{0, -\delta\} \leq \alpha - 2\theta \leq \min\{\pi/2, \pi/2 - \delta\} \quad (4)$$

4. Applying a similar reasoning to line (2,4) we get

$$\max\{0, \delta\} \leq 2\theta \leq \min\{\pi/2, \pi/2 + \delta\} \quad (5)$$

5. The flow conservation equations at buses 1 and 2 are, respectively,

$$\sin(\alpha - 2\theta) = \sin \delta + \sin \theta, \quad (6a)$$

$$\frac{1}{2.5} \sin(\alpha - 2\theta + \delta) = -\sin \delta + \frac{1}{4} \sin(2\theta - \delta) \quad (6b)$$

Let

$$\epsilon_1 \doteq \sin(\alpha - 2\theta + \delta) - \sin(\alpha - 2\theta), \quad \epsilon_2 \doteq \sin(2\theta - \delta) - \sin(2\theta).$$

Using these definitions and substituting (6a) into (6b) we obtain

$$\frac{1}{2.5} [\sin \theta + \sin \delta + \epsilon_1] = -\sin \delta + \frac{1}{4} [\sin(2\theta) + \epsilon_2], \quad \text{or} \quad (7)$$

$$\Delta(\theta) = \frac{7}{2} \sin \delta + \epsilon_1 - \frac{5}{8} \epsilon_2. \quad (8)$$

We observe that for any angle ϕ with

$$\max\{0, \pm\delta\} \leq \phi \leq \min\{\pi/2, \pi/2 \pm \delta\},$$

we have

$$|\sin(\phi) - \sin(\phi - \delta)| \leq |\delta|. \quad (9)$$

Then using (9) and (4) (or (5)) we have $|\epsilon_1| \leq \delta$ (resp., $|\epsilon_2| \leq \delta$). Hence, from (8) and (4)

$$|\Delta(\theta)| \leq 0.02563. \quad (10)$$

6. As a consequence, using Lemma 1 we have that either θ is close to zero or close to θ_0 , or more precisely

$$\text{either } 0 \leq \theta \leq 0.1057 \quad \text{or} \quad 0.578 \leq \theta \leq 0.6952 (< \pi/4). \quad (11)$$

We will refer to these two modes of operation as Mode I and II, respectively.

7. Define the throughput of the transmission system to be the total flow sent out from bus 0, which is the same as the total flow received by bus 4. Hence the throughput equals

$$\frac{1}{4} \sin(2\theta - \delta) + \sin(\theta).$$

As per observation 6 we now have that

- (I) In Mode I the throughput is less than 0.1592. The solution using $\theta = 0.1057$ and $\delta = 0$ is feasible and attains throughput greater than 0.1579.
- (II) In Mode II the throughput is at least 0.77464 and less than 0.88671. The solution using $\theta = 0.6952$ and $\delta = 0$ is feasible and attains throughput greater than 0.88648.

1.2 NP-hardness construction

Here we consider the following problem:

THROUGHPUT: Given a transmission system in the lossless AC power flow model, with a single generator and a single load, and a value $T \geq 0$, is there a feasible solution where at least T units of power are transmitted from the generator to the load?

We will show that this problem is strongly NP-hard using a reduction from one-in-three 3-SAT, defined as follows:

ONE-IN-THREE 3-SAT: Given clauses C_1, \dots, C_m on boolean variables x_1, \dots, x_n , where each C_i uses three literals, is there a truth assignment to the x_j where in each clause there is precisely one true literal?

The construction will rely on several adaptations of the transmission system B considered in the previous section. In what follows, we set

$$S \doteq 0.1592, \quad H \doteq 0.8864.$$

We assume we are given an instance in one-in-three 3SAT with notation as above. For each variable x_j , we first construct a “variable” network $V(j)$ using two copies of the B network, denoted B_j and \bar{B}_j , respectively plus three additional buses, s_j , \bar{s}_j and t_j .

There is a line connecting bus s_j (resp., \bar{s}_j) to the copy of bus 0 in B_j (\bar{B}_j), denoted by 0_j ($\bar{0}_j$). Each of the two copies of bus 4 (denoted 4_j and $\bar{4}_j$, respectively) is connected by a line to bus t_j . See Figure 3. The lines connecting s_j , \bar{s}_j and t_j to the B networks have unit reactance and very large limit.

Next, for each clause C_i we construct a “clause” network $C(i)$. Suppose $C_i = (p \vee q \vee r)$. The network $C(i)$ will contain three copies of the B network, denoted $B_{p,i}$, $B_{q,i}$ and $B_{r,i}$. See Figure 4.

The copy of bus 0 in $B_{p,i}$ is labeled $0_{p,i}$ and likewise with the other copies of bus 0 and the copies of bus 4. There are additional buses $s_{p,i}$, $s_{q,i}$ and $s_{r,i}$, connected to $0_{p,i}$, $0_{q,i}$, and $0_{r,i}$, respectively, and a bus T_i connected to $4_{p,i}$, $4_{q,i}$, and $4_{r,i}$. These new lines have unit reactance and large limit.

The variable networks and clause networks are assembled together as follows. (a) There is an additional bus, D . Bus D is connected to all buses t_j ($1 \leq j \leq n$) with lines with reactance $1/2$ and limit $S + H$. Bus D is also connected to all buses T_i ($1 \leq i \leq m$) with lines with reactance $1/2$ and limit $2S + H$. See Figure 5.

(b) For each variable x_j ($1 \leq j \leq n$) we have $C_i = (p \vee q \vee r)$. four additional buses, denoted L_j , R_j , \bar{L}_j and \bar{R}_j respectively. Bus L_j (resp., R_j) is connected to 0_j (4_j), and, for any clause C_i containing x_j , bus L_j (resp., R_j) is connected to $0_{x_j,i}$ ($4_{x_j,i}$). Likewise, bus \bar{L}_j (resp., \bar{R}_j) is connected to $\bar{0}_j$ ($\bar{4}_j$), and, for any clause C_i containing \bar{x}_j , bus \bar{L}_j (resp., \bar{R}_j) is connected to $\bar{0}_{x_j,i}$ ($\bar{4}_{x_j,i}$). All lines mentioned here have limit $1/20$ and unit reactance. See Figure 6 for an example.

(c) Bus D is the only load. For $1 \leq j \leq n$, buses s_j and \bar{s}_j are generators, each with capacity $S + H$. $1 \leq i \leq m$, let $C_i = (p \vee q \vee r)$. Then buses $s_{p,i}$, $s_{q,i}$ and $s_{r,i}$ are generators, each with capacity $2S + H$.

The following two lemmas establish the NP-hardness result.

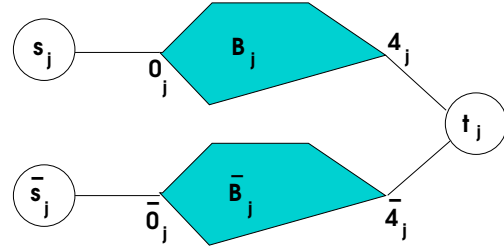


Figure 3: Network $V(j)$

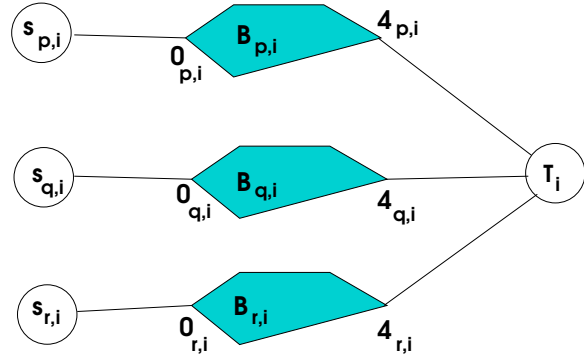


Figure 4: Network $C(i)$ corresponding to clause

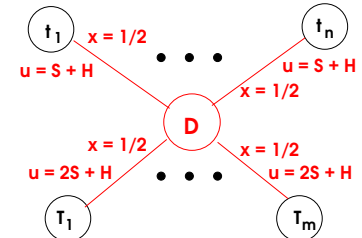


Figure 5: Attaching bus D

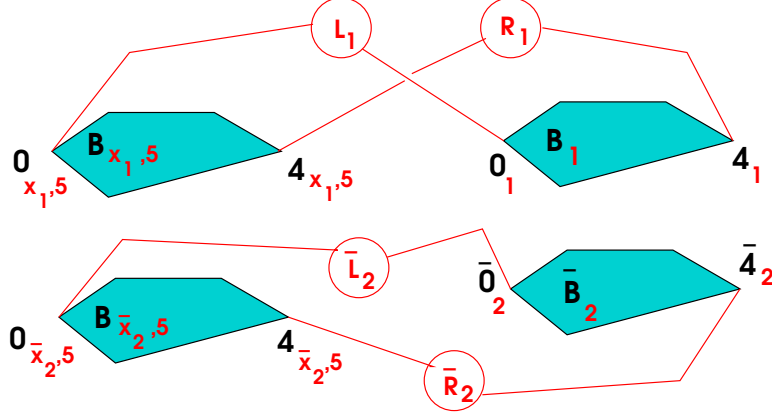


Figure 6: Assembly in the case $C_5 = (x_1 \vee \bar{x}_2 \vee x_3)$.

Lemma 2 Suppose there is a feasible solution where the total demand consumed at node D is at least $n(S+H) + m(2S+H)$. Then the instance of one-in-three 3SAT is satisfiable.

Proof. First we note that each of the subnetworks B_j, \bar{B}_j (for each variable x_j) and $B_{w,i}$ (for each clause C_i) must operate in Mode I or II as detailed above. Further, by choice of the limits on the lines connected to bus D it follows that each such line is operating at its limit. It follows that (a) for $1 \leq j \leq n$ one of B_j and \bar{B}_j operates in Mode I and the other in Mode II, and (b) for $1 \leq i \leq m$ if $C_i = (p \vee q \vee r)$ then two of $B_{p,i}, B_{q,i}$ and $B_{r,i}$ operate in Mode I and the remaining one in Mode II.

Further, suppose x_j is one of the literals in clause C_i . Since the limits of lines $(0_j, L_j)$ and $(L_j, 0_{x_j,i})$ are $1/20$, and both lines have reactance $1/2$, it follows that the absolute value of the difference of phase angles at 0_j and $0_{x_j,i}$ is at most $2 \sin^{-1}(1/10) < 0.201$. Likewise, the absolute value of the difference of phase angles at 4_j and $4_{x_j,i}$ is at less than 0.201 . Hence B_j operates in Mode I (or Mode II) if and only if $B_{x_j,i}$ operates in Mode I (Mode II, respectively).

Similarly, if \bar{x}_j is one of the literals in clause C_i , then \bar{B}_j operates in Mode I (or Mode II) if and only if $B_{x_j,i}$ operates in Mode I (Mode II, respectively).

The result is now established by using the truth assignment $x_j = \text{true}$ iff B_j operates in Mode II. ■

Lemma 3 Suppose the instance of one-in-three 3SAT is satisfiable. Then there is a feasible solution where the total demand consumed at node D is at least $n(S+H) + m(2S+H)$.

Proof. Similar to that of Lemma 2. For $1 \leq j \leq n$ we operate B_j operate in Mode II iff $x_j = \text{true}$, and for each clause $C_i = (p \vee q \vee r)$, if, say $p = x_j$ then we operate $B_{x_j,i}$ in the same mode as B_j and with buses $L_j, 0_j$ and $0_{x_j,i}$ at the same phase angle and buses $R_j, 4_j$ and $4_{x_j,i}$ at the same phase angle. And if $p = \bar{x}_j$ then we operate $\bar{B}_{x_j,i}$ in the same mode as \bar{B}_j and with a corresponding setting of phase angles. This setting of phase angles yields a feasible solution with the desired throughput, where all lines incident with buses L_j and R_j carry zero flow. ■

1.3 Membership in NP

The above proof shows that testing feasibility for a system of type (1) is an NP-hard problem. To prove NP-completeness we would need to argue for membership in NP. A straightforward proof of such a fact, if true, is unlikely, for the reason that in a feasible solution very likely the f_{ij} (and possibly even some of the θ_i) would be irrational values.

We conjecture that an approximate version of system (1) where equation (1b) is replaced with $|\sin(\theta_i - \theta_j) - x_{ij}f_{ij}| \leq \epsilon$ (where $\epsilon > 0$ is part of the input) may belong to NP.

References

- [1] G. ANDERSSON, *Modelling and Analysis of Electric Power Systems*, Power Systems Laboratory, ETH Zürich, 2004.
- [2] R. BENT, D. BIENSTOCK, AND M. CHERTKOV, *Synchronization-aware and algorithm-efficient chance constrained optimal power flow*, in Proceedings 2013 IREP Symposium-Bulk Power System Dynamics and Control, 2013.
- [3] A. BERGEN AND V. VITTAL, *Power Systems Analysis*, Prentice-Hall, 1999.
- [4] D. BIENSTOCK AND G. MUÑOZ, *LP approximations to mixed-integer polynomial optimization problems*. arXiv:1501.00288, 2014.
- [5] S. BOYD AND L. VANDENBERGHE, *Additional exercises for convex optimization, sec. 16.6*, http://www.stanford.edu/~boyd/cvxbook/bv_cvxbook_extra_exercises.pdf, (2012).
- [6] J. D. GLOVER, M. S. SARMA, AND T. J. OVERBYE, *Power System Analysis and Design*, CENGAGE Learning, 2012.
- [7] J. LAVAEI AND S. LOW, *Zero duality gap in optimal power flow problem*, IEEE Trans. Power Systems, 27 (2012), pp. 92–107.
- [8] K. LEHMANN, A. GASTIEN, AND P. HENTENRYCK, *AC-Feasibility on Tree Networks is NP-Hard*, IEEE Trans. Power Systems, 31 (2016), pp. 798–801.
- [9] A. PINAR, J. MEZA, V. DONDE, AND B. LESIEUTRE, *Optimization strategies for the vulnerability analysis of the electric power grid*, SIAM J. Optimization, 20 (2010), pp. 1786–1810.
- [10] A. VERMA, *Power grid security analysis: An optimization approach*, PhD thesis, Columbia University, 2009.

Tue, Dec. 22, 194809, 2015@rockadoodle